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Patterns of convection in spherical shells

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The problem of the pattern of motion realized in a convectively unstable system with spherical symmetry can be considered without reference to the physical details of the system. Since the solution of the linear problem is degenerate because of the spherical homogeneity, the nonlinear terms must be taken into account in order to remove the degeneracy. The solvability condition leads to the selection of patterns distinguished by their symmetries among spherical harmonics of even order. It is shown that the corresponding convective motions set in as subcritical finite amplitude instabilities.

1. Introduction

Next to the famous Rayleigh–Bénard problem of convection in a layer heated from below, the problem of convection in a spherical shell heated from within has attracted the most widespread attention among geophysicists and astrophysicists interested in convective processes. An extensive review of early work on this problem is included in the form of a special chapter in Chandrasekhar's (1961) monograph. Chandrasekhar also discusses the main motivation for the interest in the problem, namely the hypothesis of convection in the earth's mantle. Within the past decade the concept of mantle convection has evolved from a speculative idea to a well-documented phenomenon. While the details of the convective motion and the motion of the buoyancy forces have remained obscure, the revolutionary geological concept of 'plate tectonics' has given a strong boost to investigations of convection in the earth's mantle. Although a large number of papers on this subject have appeared in recent years not much attention has been paid to the question of the pattern of convection realized. In this paper we demonstrate that this aspect can be separated from other aspects of the problem and that a rather general answer to the question can be obtained.

Convection in a layer with spherical symmetry shares with the problem of convection in a plane layer of infinite extent the property that the solutions of the linearized equations are degenerate. This means that solutions with different spatial dependences correspond to the same eigenvalue and thus are equally likely to be established. This is a reflexion of the high symmetry exhibited by the spherical as well as the plane layer. The degeneracy of the linear problem is an unphysical result, however, since even minute additional effects will in general remove, or at least reduce, this degeneracy. Predominant among the effects neglected in the linear treatment of the problem are the nonlinear terms in the basic equations. In the case of convection in the earth's mantle it could be argued that the axis of rotation introduces a distinguished direction. It has been pointed out, however, that the relevant Taylor number is much too low to be significant in the problem (Runcorn 1965; McKenzie 1968). In the case of the solar convection zone, on the other hand, the Taylor number is significant even if a turbulent eddy viscosity is assumed. In an earlier paper (Busse 1970) it was shown that the Coriolis force removes the degeneracy and leads to a preference for convection in the equatorial belt.

In this paper we do a similar analysis to investigate the influence of the nonlinear terms on the problem of degeneracy. Since the nonlinear terms do not introduce preferred directions and since the physical conditions of the problem are still spherically symmetric, the problem is more involved than in the case of a slowly rotating sphere. Some guidance will be provided by analogy with the problem of convection in a plane layer. The general problem of the pattern realized in this case was considered by Schlüter, Lortz & Busse (1965) and by Busse (1962, 1967). It was found that small asymmetries in physical properties between the upper and lower parts of the layer favour the onset of convection in the form of hexagonal cells while convection in the form of rolls replaces the hexagons when the amplitude of convection becomes sufficiently large relative to the magnitude of the asymmetries. The tendency of asymmetries to favour hexagons was also found by Palm (1960) and Segel (1965). Hexagonal convection has the special property that it can exist for a limited range of finite amplitudes at Rayleigh numbers below the critical value determined by the linear theory. The difference between the lowest Rayleigh number at which hexagonal convection can exist and the critical value initially increases quadratically with the magnitude of the asymmetries. Hence the consideration of the nonlinear terms favouring hexagons may also provide an important correction to the linear stability result.

The analogue of hexagonal convection in the case of a spherical shell has not yet been investigated. Most mathematical treatments of finite amplitude convection have been restricted to two-dimensional analysis by assuming axisymmetry. A notable exception is the work by Young (1974), who computed some cases of non-axisymmetric convection. In this paper we shall develop a general theory of convection patterns in spherical shells without taking into account in detail the physical properties of the shell. Hence quantitative aspects of the problem will not be resolved by the theory. We anticipate, however, that the effects which favour hexagons in the case of a plane layer will be even more important in the spherical case. In addition to asymmetric properties the geometric asymmetry between inner and outer parts of the shell plays an important role. In planetary applications of the problem the temperature dependence of properties such as viscosity is likely to be even more important. In order to exhibit most clearly the general nature of the theory we shall not consider the general case, which may include compressibility and non-Newtonian fluid properties. Instead we shall start in § 2 with the formulation of the basic problem in terms of the Boussinesq approximation to the equations of motion, which has traditionally been used for the investigation of convection in spherical shells.

The analysis of this paper will be focused on the solvability condition which results when nonlinear effects are taken into account as a perturbation of the linear problem. In § 3 we shall discuss the general implications of the solvability condition and demonstrate its equivalence to a simple extremum problem. In spite of the simplicity of its formulation the problem cannot be solved in general. By considering it for different orders l of the spherical harmonics which describe the solution of the linear problem we obtain solutions for $l = 2$, 4 and 6 in §§ 4, 5 and 6, respectively. Only partial solutions can be obtained in the cases of larger values of l, which are considered in § 7; some of the remaining unresolved mathematical questions are mentioned at that point. In § 8 the physical implications of the results are discussed. Unfortunately no direct relation between the results and the problem of mantle convection can be seen from the present observational evidence. Apart from this original motivation for the problem the basic nature of results demonstrating the importance of non-axisymmetric patterns will make the present results relevant in a number of other applications.

2. Mathematical formulation

We consider the problem of convection in a homogeneous fluid contained between two concentric spherical boundaries with radii $r_0 h$ and $(r_0+1)h$. We assume a spherically symmetric gravity force and a distribution of heat sources of the same symmetry within the fluid or in the core below it. Thus a static equilibrium exists for all values of the temperature difference ΔT between the boundaries. If the temperature increases in the direction of gravity the static equilibrium becomes unstable when ΔT exceeds a finite value. The buoyancy force overcomes the stabilizing effects of viscous and thermal dissipation in this case and convective motions set in.

Even though it is not necessary for the purpose of this paper to do so, we assume the equations of motions in the Boussinesq approximation. The properties of the fluid are regarded as constants with the exception of the temperature dependence of the density, which is taken into account in the gravity term only. It will become obvious from the analysis that the results apply to the most general types of fluids as long as the spherical symmetry is preserved.

It is convenient to introduce dimensionless variables by using the thickness h of the fluid shell, h^2/κ and ΔT as scales for length, time and temperature, respectively, where κ denotes the thermal diffusivity. Accordingly, the equations of motion for the velocity vector $\mathbf{u}$ and the heat equation for the deviation Θ of the temperature from the static temperature field $T(r)$ are given by

$$-\nabla\times(\nabla\times\mathbf{u})+R\Theta\mathbf{r}\gamma(r)-\nabla\pi = (\partial\mathbf{u}/\partial t-\mathbf{u}\times(\nabla\times\mathbf{u}))P^{-1}, \tag{2.1}$$

$$\nabla.\mathbf{u} = 0, \tag{2.2}$$

$$\nabla^2\Theta+\mathbf{u}.\mathbf{r}T'(r)/r = \partial\Theta/\partial t+\mathbf{u}.\nabla\Theta. \tag{2.3}$$

The term $\nabla\pi$ includes the pressure and other terms which can be written in the form of a gradient. $\mathbf{r}$ denotes the position vector with respect to the centre of the shell. The r dependence of the gravity force $-g_0\gamma(r)\mathbf{r}$ has been normalized in such a way that g_0 represents the acceleration due to gravity at the inner boundary $r = r_0$ of the shell. The Rayleigh number R and Prandtl number P are defined by

$$R = \beta g_0 \Delta T h^3/\kappa\nu, \quad P = \nu/\kappa,$$

where β denotes the coefficient of thermal expansion and ν is the kinematic viscosity.

To eliminate the equation of continuity (2.2) we use the following general representation for the solenoidal vector field $\mathbf{u}$ in terms of poloidal and toroidal fields:

$$\mathbf{u} = \nabla\times(\nabla\times\mathbf{r}\Phi)+\nabla\times\mathbf{r}\Psi. \tag{2.4}$$

By operating on (2.1) with $\mathbf{r}.\nabla\times(\nabla\times\quad)$ we obtain

$$(\nabla^2-\partial/\partial t)\nabla^2L^2\Phi - R\gamma(r)L_2\Theta = P^{-1}\mathbf{r}.\nabla\times\{\nabla\times[\mathbf{u}\times(\nabla\times\mathbf{u})]\}, \tag{2.5}$$

where L^2 is the negative two-dimensional Laplacian on the unit sphere, i.e. in spherical co-ordinates (r,θ,ϕ)

$$L^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{2.6}$$

Since the linear part of (2.3) involves only Φ we need not consider the variable Ψ, at least as far as the linear problem is concerned. In fact, the equation for Ψ admits only decaying solutions when nonlinear terms are neglected, as is shown in Chandrasekhar's (1961) book.

We shall consider an expansion for Φ, Θ and R in powers of the amplitude ϵ of convection:

$$\Phi = \epsilon\Phi^{(1)}+\epsilon^2\Phi^{(2)}+\ldots, \quad \Theta = \epsilon\Theta^{(1)}+\epsilon^2\Theta^{(2)}+\ldots, \quad R = R^{(0)}+\epsilon R^{(1)}+\ldots.$$

From (2.3) and (2.5) to order ϵ we obtain the well-known linear problem

$$\nabla^4L^2\Phi^{(1)} - R^{(0)}\gamma(r)L^2\Theta^{(1)} = 0, \tag{2.7a}$$

$$\nabla^2\Theta^{(1)} + T'(r)r^{-1}L^2\Phi^{(1)} = 0. \tag{2.7b}$$

We have assumed steady convection since it has been shown by Chandrasekhar (1961), at least in special cases, that oscillatory convection cannot occur. Because of the spherical homogeneity of the problem the general solution of (2.7) can be written in the form

$$\Phi^{(1)} = f(r)w_l(\theta,\phi), \quad \Theta^{(1)} = g(r)w_l(\theta,\phi), \tag{2.8a}$$

where w_l is defined by

$$w_l(\theta,\phi) = \sum_{m=0}^{l}(\alpha_m\cos m\phi+\beta_m\sin m\phi)\hat{P}_l^m(\cos\theta). \tag{2.8b}$$

The functions $\hat{P}_l^m(\cos\theta)$ differ by a factor from the commonly used associated Legendre functions,

$$\hat{P}_l^m(x) \equiv [(4l+2)(1-\tfrac{1}{2}\delta_{m0})(l-m)!/(l+m)!]^{\frac{1}{2}}P_l^m(x),$$

with the result that the average of the square of each spherical harmonic over the spherical surface is unity:

$$\langle|\hat{P}_l^m(\cos\theta)\cos m\phi|^2\rangle = \langle|\hat{P}_l^m(\cos\theta)\sin m\phi|^2\rangle = 1$$

for $m = 0, 1, \ldots, l$. For later use we shall define the average in such a way that it can be used for radially dependent functions as well:

$$\langle \;\; \rangle = \frac{1}{4\pi[r_0(r_0+1)+\frac{1}{3}]}\int_{r_0}^{r_0+1}\int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\phi \, r^2 \, dr. \tag{2.9}$$

Since w_l satisfies the relation

$$L^2 w_l = l(l+1)\, w_l$$

(2.7) can be reduced to an ordinary differential equation for $f(r)$ after $g(r)$ has been eliminated:

$$\left[\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}-\frac{l(l+1)}{r}\right)\frac{1}{\gamma(r)}\left(\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}-\frac{l(l+1)}{r^2}\right)^2+R^{(0)}T'(r)/r\right]f = 0. \tag{2.10}$$

Together with the appropriate boundary conditions (2.10) represents a homogeneous eigenvalue problem with eigenvalue $R^{(0)}$. Of particular interest is the lowest eigenvalue $R^{(0)}$ as a function of l, which determines the limit of static stability with respect to infinitesimal disturbances in the form of spherical harmonics of order l.

In this paper we shall not consider special solutions of (2.10). Instead we focus our attention on the degeneracy of the linear eigenvalue problem, which manifests itself in the fact that the $2l+1$ coefficients α_n and β_n in (2.8 *b*) remain undetermined. Since $R^{(0)}$ does not depend on m, the linear problem has $2l+1$ linear independent eigensolutions. The purpose of the following analysis is to find out how the degeneracy of the linear problem is removed or at least reduced by the consideration of nonlinear effects.

From (2.3) and (2.5) at order ϵ^2 we obtain

$$\nabla^4 L^2 \Phi^{(2)} - R^{(0)}\gamma(r)\, L^2\Theta^{(2)}$$
$$= -P^{-1}L^2[\nabla(\mathbf{r}.\nabla+1)\,\Phi^{(1)}.\nabla\nabla^2\Phi^{(1)} - \nabla^2\Phi^{(1)}\mathbf{r}.\nabla\nabla^2\Phi^{(1)}] + R^{(1)}\gamma(r)\, L^2\Theta^{(1)}, \tag{2.11 a}$$

$$\nabla^2\Theta^{(2)} + T'(r)\, L^2\Phi^{(2)} r^{-1} = [\nabla\times(\nabla\times\mathbf{r}\Phi^{(1)})].\nabla\Theta^{(1)}. \tag{2.11 b}$$

This inhomogeneous system of linear equations has a solution if and only if the inhomogeneous part is orthogonal to all solutions (Φ^+, Θ^+) of the adjoint homogeneous system of equations

$$\nabla^4 L^2\Phi^+ - R^{(0)}T'(r)\, r^{-1}L^2\Theta^+ = 0, \tag{2.12 a}$$

$$\nabla^2\Theta^+ + \gamma(r)\, L^2\Phi^+ = 0. \tag{2.12 b}$$

The solutions of this system of equations together with the corresponding adjoint boundary conditions can obviously be written in the same form (2.8) as the solutions of the first-order problem:

$$\Phi^+ = f^+(r)\, w_l^+(\theta,\phi), \quad \Theta^+ = g^+(r)\, w_l^+(\theta,\phi), \tag{2.13}$$

where w_l^+ represents the general spherical harmonic of order l as in definition (2.9). The solvability condition is obtained by multiplying (2.11 *a*) by Φ^+ and (2.11 *b*) by $-R^{(0)}\Theta^+$, adding the results and averaging over the fluid shell. By definition the left-hand side vanishes after partial integrations have been performed and the right-hand side yields

$$0 = R^{(1)}\langle\gamma\Phi^{+}L^{2}\Theta^{(1)}\rangle - P^{-1}\langle\Phi^{+}L^{2}[\nabla(\mathbf{r}.\nabla+1)\,\Phi^{(1)}.\nabla\nabla^{2}\Phi^{(1)} - \nabla^{2}\Phi^{(1)}\mathbf{r}.\nabla\nabla^{2}\Phi^{(1)}]\rangle$$
$$- R^{(0)}\langle\Theta^{+}(\nabla(1+\mathbf{r}.\nabla)\,\Phi^{(1)} - \mathbf{r}\nabla^{2}\Phi^{(1)}).\nabla\Theta^{(1)}\rangle, \quad (2.14)$$

where the angular brackets indicate averages over the fluid shell according to definition (2.9).

Since solutions (2.8) and (2.13) vary only in their θ and ϕ dependences at a given value of l and since the r dependence is not of particular interest we rewrite expression (2.14) in the form

$$R^{(1)}\langle w_l^{+}w_l\rangle - \langle w_l^{+}w_l w_l\rangle M(l) = 0. \quad (2.15)$$

In deriving (2.15) we have used our freedom to normalize the linear solution by setting

$$\langle\gamma(r)f^{+}(r)\,l(l+1)\,g(r)\rangle \equiv 1. \quad (2.16)$$

A typical example of the partial integrations required for the derivation of (2.15) is

$$\begin{aligned}\langle w_l^{+}\nabla_s w_l.\nabla_s w_l(g^{+}(rf)'g)\rangle &= \langle w_l^{+}w_l w_l l(l+1)\,r^{-2}(g^{+}(rf)'g)\rangle \\ &\quad - \langle w_l\nabla_s w_l^{+}.\nabla_s w_l(g^{+}(rf)'g)\rangle \\ &= \langle w_l\nabla_s w_l^{+}.\nabla_s w_l^{-}(g^{+}(rf)'g)\rangle \\ &= \tfrac{1}{2}\langle w_l^{+}w_l w_l l(l+1)\,r^{-2}(g^{+}(rf)'g)\rangle, \end{aligned} \quad (2.17)$$

where $$\nabla_s = \left(0, \frac{1}{r}\frac{\partial}{\partial\theta}, \quad \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

represents the surface components of the gradient.

Before we continue the discussion of the solvability condition (2.15) we note that the linear problem (2.7) is self-adjoint in the special case when

$$\frac{d}{dr}\frac{r\gamma(r)}{T'(r)} = 0 \quad (2.18)$$

and when in addition the boundary conditions satisfy the condition for self-adjointness, as was pointed out by Joseph & Carmi (1966). The function $M(l)$ vanishes for all l in the self-adjoint case, as can be readily seen from the particular form of the nonlinear advective terms in the basic equations (2.1) and (2.3). The energy stability analysis of Joseph & Carmi (1966) may in fact be used to obtain an upper bound for $M(l)$. For the purpose of the present paper we have in mind a much more general system of basic equations than (2.1)–(2.3), which we have introduced in order to formulate the problem in a simple model case. The consideration of temperature-dependent fluid properties, for instance, leads to additional nonlinear terms which can be taken into account in the stability condition (2.15) in the same way as the terms in expression (2.14). The corresponding contributions to $M(l)$ would not vanish in this case even if the linear problem happened to be self-adjoint. For this reason we may regard the possibility of vanishing $M(l)$ as exceptional and proceed with the general problem posed by condition (2.15).

3. The solvability condition

The solvability condition (2.15) is equivalent to the following system of $2l+1$ equations which represent the $2l+1$ independent choices that can be made for w_l^+:

$$R^*\alpha_m = \langle \hat{P}_l^m(\cos\theta)\cos m\phi w_l w_l\rangle, \quad m = 0, 1, \ldots, l, \tag{3.1 a}$$

$$R^*\beta_m = \langle \hat{P}_l^m(\cos\theta)\sin m\phi\, w_l w_l\rangle, \quad m = 1, \ldots, l. \tag{3.1 b}$$

To simplify the notation we have introduced $R^* \equiv R^{(1)}/M(l)$. In addition we need a normalization condition, which we shall specify in the form

$$\langle w_l w_l\rangle = \alpha_0^2 + \sum_{m=1}^{l} (\alpha_m^2 + \beta_m^2) = 1. \tag{3.1 c}$$

Relations (3.1) represent a system of $2l+2$ nonlinear inhomogeneous equations for the $2l+2$ unknowns α_n, β_n and R^*. In the following we shall solve this system of equations in special cases.

Before entering the detailed analysis it is of interest to note some general properties of the system (3.1). It may readily be seen that (3.1) are the Euler equations for the stationary value R^* of the functional

$$\mathscr{R}(w_l) \equiv \langle w_l w_l w_l\rangle \tag{3.2}$$

under the side constraint $\langle w_l w_l\rangle = 1$. As we shall point out later, the absolute extremum R^* of the functional (3.2) is of particular interest.

Another general property of (3.1) is the fact that they are satisfied by $R^* = 0$ for all values of α_n and β_n restricted only by the normalization condition (3.1 c) when l is an odd integer. This follows from the property of any spherical harmonic that its values at opposite points on the sphere are equal, with the same or opposite sign depending on whether l is even or odd:

$$w_l(\phi, \theta) = (-1)^l w_l(\phi+\pi, \pi-\theta).$$

Thus
$$\langle w_l w_l w_l\rangle = 0 \quad \text{for odd } l$$

since the symmetric function w_l^2 is multiplied by an antisymmetric function.

The absolute value of R^* is always less than or equal to unity, since the general variational problem

$$\mathscr{R}(w) \equiv \langle www\rangle/\langle ww\rangle^{\frac{3}{2}} \tag{3.3}$$

has the solution $w \equiv \pm 1$, corresponding to $R^* = \pm 1$. This is the solution in the case $l = 0$, which is not of physical interest since it does not correspond to convective motion. There does not seem to exist a general method for solving (3.1) in the case of even l. Hence we shall consider cases of different l separately, starting with $l = 2$, which is the only case in which a complete solution can be obtained in analytical form. For higher values of l we shall use the fact that an extremalizing solution exhibits in general a high degree of symmetry. In particular we shall find that the extremalizing solutions in the cases $l = 4$ and 6 exhibit the symmetry of the Platonic bodies, with the exception of the tetrahedron.

A difficulty in solving (3.1) is that for any solution there exists a continuum of solutions obtained by rotations of the co-ordinate system from the given solution.

Part of this indeterminacy can be eliminated by suitable additional conditions. For instance, we can assume $\beta_1 = 0$ without losing generality, since any solution satisfies this condition after a suitable rotation of the co-ordinate system about the polar axis. For reasons of symmetry it is likely that all extremalizing solutions possess a plane of symmetry through the origin, with the consequence that they can be written in a form with $\beta_n = 0$ for $n = 1, \ldots, l$. However we have not found a proof that this hypothesis is correct.

For the evaluation of the integrals on the right-hand sides of (3.1 a, b) we introduce the notation

$$A^l_{npq} \equiv \langle \hat{P}^n_l(\cos\theta)\,\hat{P}^p_l(\cos\theta)\,\hat{P}^q_l(\cos\theta)\cos n\phi\cos p\phi\cos q\phi\rangle.$$

Most of the coefficients A^l_{npq} vanish. Others can be evaluated from the formulae

$$A^l_{000} = (2l+1)^{\frac{3}{2}}\frac{[(\frac{3}{2}l)!]^2\,[l!]^3}{(3l+1)!\,[(\frac{1}{2}l)!]^6},$$

$$A^l_{0mm} = (2l+1)^{\frac{3}{2}}\frac{(-1)^{\frac{1}{2}l}\,[l!]^2\,(\frac{3}{2}l)!}{(3l+1)!\,[(\frac{1}{2}l)!]^3}\sum_{t=0}^{l-m}(-1)^t\frac{(l+m+t)!\,(2l-m-t)!}{(l-m-t)!\,(m+t)!\,(l-t)!\,t!},$$

$$A^l_{l\frac{1}{2}l\frac{1}{2}l} = 1/2^{\frac{1}{2}}\,(2l+1)^{\frac{3}{2}}\frac{l!\,[2l!]^{\frac{1}{2}}\,[(\frac{3}{2}l)!]^2}{[(\frac{1}{2}l)!]^4\,(3l+1)!},$$

which have been obtained from corresponding formulae given by Gaunt (1929). General formulae for other triple integrals of spherical harmonics have been discussed in the literature. We refer to the most recent paper on this topic, by Kaula (1975).

A special solution of (3.1) is the axisymmetric solution

$$\alpha^{(l)}_0 = 1, \quad \alpha^{(l)}_m = \beta^{(l)}_m = 0 \quad \text{for} \quad n = 1, \ldots, l. \tag{3.4a}$$

The corresponding value of R^* is given by

$$R^{*(l)}_0 = A^l_{000}. \tag{3.4b}$$

It is not surprising that the axisymmetric pattern of convection satisfies the solvability condition. We shall see, however, that in general the axisymmetric solution is not the one realized physically since other solutions yield larger values of R^*. It is obvious from the symmetry of (3.1) that for any solution $(R^*, w_l) \not\equiv 0$ there exists a solution $(-R^*, -w_l)$. Since the expansion parameter ϵ can assume either sign we do not loose generality if we restrict our attention to solutions with positive R^*, as we have already done in the case of the axisymmetric solution (3.4).

4. The case $l = 2$

Using the following values of the triple integrals,

$$A_{000} \equiv A = \tfrac{2}{7}\times 5^{\frac{1}{2}}, \quad A_{011} = \tfrac{1}{2}A, \quad A_{022} = -A, \quad A_{112} = \tfrac{1}{2}\times 3^{\frac{1}{2}}A,$$

the system of equations (3.1) can be written in the form

$$R^*\alpha_0 = \mathrm{A}(\alpha_0^2 + \tfrac{1}{2}\alpha_1^2 + \tfrac{1}{2}\beta_1^2 - \alpha_2^2 - \beta_2^2), \tag{4.1a}$$

$$R^*\alpha_1 = A[\alpha_0\alpha_1 + 3^{\frac{1}{2}}(\alpha_1\alpha_2 + \beta_1\beta_2)], \tag{4.1b}$$

$$R^*\beta_1 = A[\alpha_0\beta_1 + 3^{\frac{1}{2}}(\alpha_1\beta_2 - \alpha_2\beta_1)], \tag{4.1 c}$$

$$R^*\alpha_2 = A[-2\alpha_0\alpha_2 + \tfrac{1}{2}\times 3^{\frac{1}{2}}(\alpha_1^2 - \beta_1^2)], \tag{4.1 d}$$

$$R^*\beta_2 = A(-2\alpha_0\beta_2 + 3^{\frac{1}{2}}\alpha_1\beta_2), \tag{4.1 e}$$

$$1 = \alpha_0^2 + \alpha_1^2 + \beta_1^2 + \alpha_2^2 + \beta_2^2. \tag{4.1 f}$$

Assuming $\beta_1 = 0$ we find from (4.1 *c*) that β_2 vanishes as well. The remaining four equations can be solved explicitly. The general solution is

$$\alpha_1^2 = \tfrac{2}{3}(1+2\alpha_0)(1-\alpha_0), \quad \alpha_2 = (1-\alpha_0)/3^{\frac{1}{2}}, \quad R^* = A \tag{4.2}$$

for arbitrary values of α_0 with $|\alpha_0| \leqslant 1$. Comparison with (3.4) shows that solution (4.2) represents the axisymmetric solution rotated about the axis $\theta = \frac{1}{2}\pi$, $\phi = \pm\frac{1}{2}\pi$. Hence the axisymmetric solution is the only solution in the case $l = 2$ which satisfies the solvability condition (2.15). This is a non-trivial result, since the general representation (2.8 *b*) contains other patterns, which are excluded by the solvability condition. We emphasize this point in contrast to the case $l = 1$, where all possible choices of coefficients in (2.8 *b*) correspond to transformations of the same solution. For this reason we do not encounter the problem of degeneracy for $l = 1$.

5. The case $l = 4$

The general solution obtained in the case $l = 2$ suggests that solutions of (3.1) can always be obtained in a form with $\beta_m = 0$ for all m. In order to simplify the analysis we shall use this hypothesis and neglect the equations for the β_n. Thus we obtain

$$R^*\alpha_0 = A_{000}\alpha_0^2 + A_{011}\alpha_1^2 + A_{022}\alpha_2^2 + A_{033}\alpha_3^2 + A_{044}\alpha_4^2, \tag{5.1 a}$$

$$R^*\alpha_1 = 2A_{011}\alpha_0\alpha_1 + 2A_{112}\alpha_1\alpha_2 + 2A_{123}\alpha_2\alpha_3 + 2A_{134}\alpha_3\alpha_4, \tag{5.1 b}$$

$$R^*\alpha_2 = 2A_{022}\alpha_0\alpha_2 + A_{112}\alpha_1^2 + 2A_{123}\alpha_1\alpha_3 + 2A_{234}\alpha_2\alpha_4, \tag{5.1 c}$$

$$R^*\alpha_3 = 2A_{033}\alpha_0\alpha_3 + 2A_{123}\alpha_1\alpha_2 + 2A_{134}\alpha_1\alpha_4, \tag{5.1 d}$$

$$R^*\alpha_4 = 2A_{044}\alpha_0\alpha_4 + 2A_{123}\alpha_1\alpha_3 + A_{224}\alpha_2^2, \tag{5.1 e}$$

$$1 = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + \alpha_4^2. \tag{5.1 f}$$

Evaluation of the triple integrals yields

$$A_{000} \equiv A = \frac{3^5 \times 2}{7\times 11\times 13}, \quad A_{011} = \tfrac{1}{2}A, \quad A_{022} = -\tfrac{11}{18}A, \quad A_{033} = -\tfrac{7}{6}A,$$

$$A_{044} = \tfrac{7}{9}A, \quad A_{224} = \frac{(35)^{\frac{1}{2}}}{6}A, \quad A_{112} = \frac{(5)^{\frac{1}{2}}}{3}A, \quad A_{123} = \frac{(35)^{\frac{1}{2}}}{18}A, \quad A_{134} = \frac{7\times(5)^{\frac{1}{2}}}{18}A.$$

In order to obtain a general impression of the manifold of solution (5.1) an extensive search was made for extreme values of the functional (3.3) with $w = w_4$. Discrete values between ± 10 were assigned to the variables $x_m = \alpha_n/\alpha_0$, $m = 1, \ldots, 4$, and the grid system was refined in regions where the absolute value of the functional became large. This numerical scheme did not indicate any

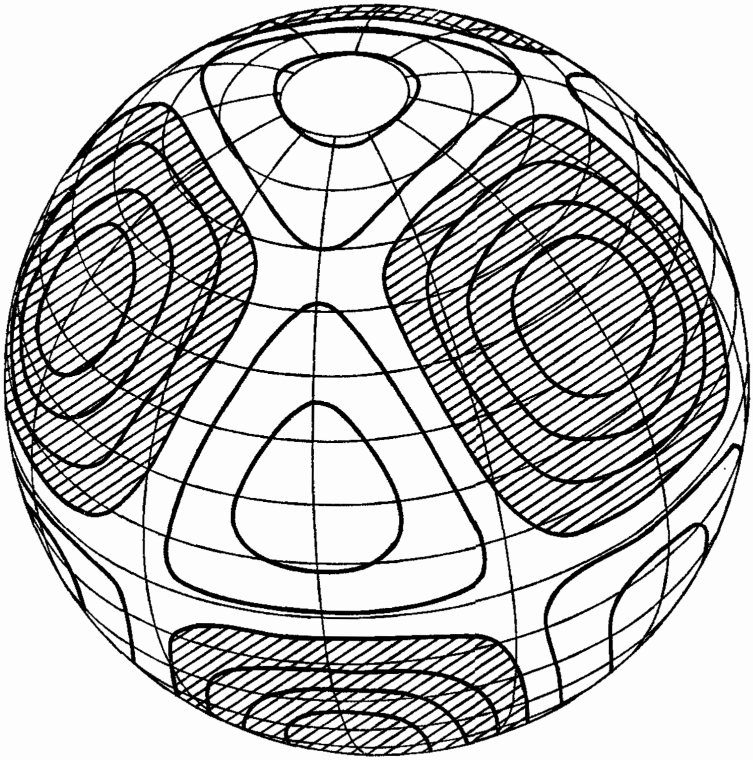

FIGURE 1. Pattern of convection with cubic symmetry in the case $l = 4$. Lines of constant radial velocity have been drawn for $w = 0{\cdot}4n(\frac{3}{7})^{\frac{1}{2}}$, $n = -3, -2, -1, 0, 1, 2$. Depending on the sign of $\epsilon R^{(1)}$ the motion is ascending or descending in the shaded areas.

solution other than transformations of the axisymmetric solution (3.4) and of the solution

$$\alpha_0 = \tfrac{1}{2}(\tfrac{7}{3})^{\frac{1}{2}}, \quad \alpha_4 = \tfrac{1}{2}(\tfrac{5}{3})^{\frac{1}{2}}, \quad \alpha_1 = \alpha_2 = \alpha_3 = 0, \tag{5.2a}$$

$$R_1^* = \tfrac{7}{9}(\tfrac{7}{3})^{\frac{1}{2}} A = \frac{18 \times (21)^{\frac{1}{2}}}{11 \times 13}. \tag{5.2b}$$

Although the numerical scheme emphasized extreme values of the functional (3.3), no indication of other stationary values was found, which suggests that (5.2*b*) and the axisymmetric value $R_0^* = A$ are the only values of R^* except for changes in sign for which the solvability condition (3.1) is satisfied in the case $l = 4$. We note that R_1^* is larger than the axisymmetric value, with the consequence that in contrast to the case $l = 2$ a non-axisymmetric solution provides the extremum of the functional (3.2).

Solution (5.2*a*) exhibits the symmetries of a cube with opposite sides facing the poles or of an octahedron with opposite vertices pointing towards the poles. This suggests another simple representation of solution (5.2) when the co-ordinate system is rotated in such a way that the axis is aligned with a largest diameter of the cube. The representation obtained in this case,

$$\alpha_0 = -(\tfrac{7}{27})^{\frac{1}{2}}, \quad \alpha_3 = 2(\tfrac{5}{27})^{\frac{1}{2}}, \quad \alpha_1 = \alpha_2 = \alpha_4 = 0, \tag{5.3}$$

has been used in figure 1, which exhibits the asymmetry between positive and negative regions of w typical of the extremalizing solutions of (3.2). Depending on the sign of ϵ these regions correspond to either ascending or descending convective motions. The cube-like or six-cell solution (5.2) appeared in the numerical computations of Young (1974), indicating the particular stability of the extremalizing solutions. We shall discuss this property in more detail in § 8 using the analogy with hexagonal convection cells in a plane layer.

6. The case $l = 6$

The increasing complexity of (3.1) for increasing l prohibits a complete investigation of the manifold of solutions for larger values of l. The number of distinct solutions increases strongly with l. Only solutions corresponding to extremal values of R^* are of physical interest, however. Since those solutions are also distinguished by their high symmetry, we shall use this property as a guide in the search for the extremalizing solution.

The system of equations (3.1) in the case $l = 6$ is sufficiently similar to (5.1) that it is not necessary to write it down explicitly. We note only a few of the triple integrals which will be needed for the calculation of solutions:

$$A_{000} \equiv A = \frac{400 \times (13)^{\frac{1}{2}}}{11 \times 17 \times 19}, \quad A_{033} = -\tfrac{43}{40}A, \quad A_{044} = -\tfrac{1}{5}A,$$
$$A_{055} = \tfrac{11}{8}A, \quad A_{066} = -\tfrac{11}{20}A, \quad A_{336} = \left(\frac{3 \times 7 \times 11}{200}\right)^{\frac{1}{2}} A.$$

The fact that the extremalizing solution in the case $l = 4$ exhibits the symmetry of a simple polyhedron suggests that the extremalizing solution for $l = 6$ also corresponds to a polyhedron. Indeed, it is easily verified that a solution reflecting the symmetries of a dodecahedron satisfies the equations

$$\alpha_0 = \tfrac{1}{5} \times (11)^{\frac{1}{2}}, \quad \alpha_5 = \tfrac{1}{5} \times (14)^{\frac{1}{2}}, \quad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_6 = 0, \tag{6.1 a}$$

$$R_1^* = \tfrac{11}{20}(11)^{\frac{1}{2}} A = \frac{20}{17 \times 19}(11 \times 13)^{\frac{1}{2}}. \tag{6.1 b}$$

The same symmetry is exhibited by an icosahedron which has faces in the form of equilateral triangles. Hence another simple representation of solution (6.1) can be obtained by using the triangular symmetry:

$$\alpha_0 = \tfrac{1}{9}(11)^{\frac{1}{2}}, \quad \alpha_3 = \frac{1}{9}\left(\frac{11 \times 14}{3}\right)^{\frac{1}{2}}, \quad \alpha_6 = -\frac{2}{9}\left(\frac{14}{3}\right)^{\frac{1}{2}}, \quad \alpha_1 = \alpha_2 = \alpha_4 = \alpha_5 = 0.$$

Lines of equal vertical velocity are shown in figure 2. The structure of ten cells exhibited by the solution resembles the hexagonal cell pattern in a plane layer with the difference that the cells are pentagons rather than hexagons.

There are two additional solutions in the case $l = 6$:

$$\alpha_0 = -(\tfrac{1}{8})^{\frac{1}{2}}, \quad \alpha_4 = (\tfrac{7}{8})^{\frac{1}{2}}, \quad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_5 = \alpha_6 = 0, \quad R_2^* = \frac{1}{5 \times 2^{\frac{1}{2}}} A \tag{6.2}$$

and

$$\alpha_0 = -(\tfrac{11}{53})^{\frac{1}{2}}, \quad \alpha_6 = (\tfrac{42}{53})^{\frac{1}{2}}, \quad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0, \quad R_3^* = \tfrac{11}{10}(\tfrac{11}{53})^{\frac{1}{2}} A. \tag{6.3}$$

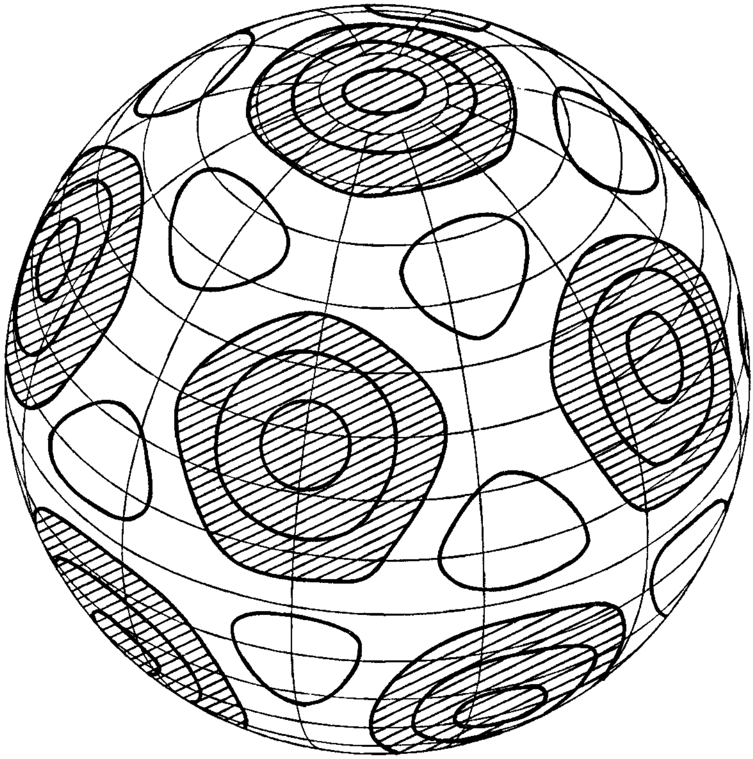

FIGURE 2. Pattern of convection with the symmetry of a dodecahedron in the case $l = 6$. Lines of constant radial velocity have been drawn for $w = 2n/(11\times 13)^{\frac{1}{2}}$, $n = -2, -1, 0, 1$. Depending on the sign of $\epsilon R^{(1)}$ the motion is ascending or descending in the shaded areas.

These solutions are not of physical interest because of their relatively low values for R^*. Even though a complete investigation of all solutions has not been made, it can hardly be doubted that the absolute extremum of R^* is attained by solution (6.1).

7. Solution for large l

The solutions in the cases $l = 4$ and $l = 6$ have exhausted the symmetries of the Platonic bodies with the exception of the tetrahedron. The symmetry of the latter body cannot be realized among the extremalizing solutions since it does not satisfy the basic symmetry of polar points common to all spherical harmonics of even order. Since none of the solutions for larger l is distinguished by its symmetry it becomes difficult to isolate the solution corresponding to the absolute extremum of R^*.

All solutions discussed so far belong to one of the two classes

$$\left.\begin{array}{l}\alpha_0 \neq 0, \quad \alpha_n, \alpha_{2n} \neq 0 \quad \text{for} \quad \tfrac{1}{3}l < n \leqslant \tfrac{1}{2}l, \\ \alpha_m = 0 \quad \text{otherwise;}\end{array}\right\} \tag{7.1 a}$$

$$\left.\begin{array}{l}\alpha_0 \neq 0, \quad \alpha_n \neq 0 \quad \text{for a single} \quad n > \tfrac{1}{2}l, \\ \alpha_m = 0 \quad \text{otherwise.}\end{array}\right\} \tag{7.1 b}$$

An inspection of (3.1) shows that solutions of the form (7.1) are always possible, and it is easy to derive the explicit expressions for the coefficients α_n. In order to calculate those solutions in the case $l = 8$ we determine first the relevant triple integrals:

$$A_{000} \equiv A = \frac{7^2 \times 10 \times (17)^{\frac{1}{2}}}{23 \times 19 \times 13}, \quad A_{033} = -\tfrac{73}{70}A, \quad A_{044} = -\tfrac{5}{14}A, \quad A_{055} = \tfrac{13}{14}A,$$

$$A_{066} = \tfrac{13}{14}A, \quad A_{077} = -\tfrac{13}{10}A, \quad A_{088} = \tfrac{13}{35}A, \quad A_{336} = \frac{(26 \times 33)^{\frac{1}{2}}}{35}A,$$

$$A_{448} = \frac{3}{14}\left(\frac{13 \times 11}{5}\right)^{\frac{1}{2}} A.$$

All possible solutions in the two classes (7.1) correspond to one of only two different values of R^*. Those corresponding to the larger value of R^* are

$$R_1^* = \frac{13(13)^{\frac{1}{2}}}{35}A, \tag{7.2}$$

with
$$\alpha_0 = (\tfrac{13}{25})^{\frac{1}{2}}, \quad \alpha_5 = (\tfrac{12}{25})^{\frac{1}{2}}, \quad \alpha_m = 0 \quad \text{otherwise;}$$

$$\alpha_0 = (\tfrac{13}{25})^{\frac{1}{2}}, \quad \alpha_6 = (\tfrac{12}{25})^{\frac{1}{2}}, \quad \alpha_m = 0 \quad \text{otherwise;}$$

$$\alpha_0 = \frac{-(13)^{\frac{1}{2}}}{7} \quad \alpha_7 = \tfrac{6}{7}, \quad \alpha_m = 0 \quad \text{otherwise;}$$

$$\alpha_0 = -\tfrac{1}{9}(13)^{\frac{1}{2}}, \quad \alpha_3 = \frac{1}{9}\left(\frac{13 \times 14}{3}\right)^{\frac{1}{2}}, \quad \alpha_6 = \frac{(22)^{\frac{1}{2}}}{9 \times 3^{\frac{1}{2}}}, \quad \alpha_m = 0 \quad \text{otherwise;}$$

or
$$\alpha_0 = \frac{(13)^{\frac{1}{2}}}{40}, \quad \alpha_4 = \frac{(21 \times 13)^{\frac{1}{2}}}{20}, \quad \alpha_8 = -\tfrac{3}{8}(\tfrac{11}{5})^{\frac{1}{2}}, \quad \alpha_m = 0 \quad \text{otherwise.}$$

The solutions corresponding to the other value of R^* are

$$R_2^* = \tfrac{1}{7}(33)^{\frac{1}{2}}A, \tag{7.3}$$

with
$$\alpha_0 = \tfrac{1}{9}(\tfrac{11}{3})^{\frac{1}{2}}, \quad \alpha_3 = \tfrac{2}{27}(70)^{\frac{1}{2}}, \quad \alpha_6 = \tfrac{4}{27}(26)^{\frac{1}{2}}, \quad \alpha_m = 0 \quad \text{otherwise}$$

or
$$\alpha_0 = \tfrac{1}{8}(33)^{\frac{1}{2}}, \quad \alpha_4 = \tfrac{1}{4}(\tfrac{7}{3})^{\frac{1}{2}}, \quad \alpha_8 = \tfrac{1}{8}(\tfrac{65}{3})^{\frac{1}{2}}, \quad \alpha_m = 0 \quad \text{otherwise.}$$

Obviously, not all solutions corresponding to R_1^* are transformations of each other. In cases when distinct solutions share the same value of R^* the degeneracy is not removed and higher-order effects must be considered.

In contrast to the cases $l = 4$ and $l = 6$, where strong arguments support our contention that the absolute extremum of R^* is given by (5.2*b*) and (6.1*b*), respectively, we do not have similar arguments to prove that the value (7.2) is indeed the largest possible value in the case $l = 8$. Nevertheless, it seems likely to us that this is indeed the case since we have exhausted the possibilities for obtaining simple solutions. In figure 3 extremalizing values of R^* are shown in comparison with those corresponding to axisymmetric solutions. While the axisymmetric value shows a monotonic decay for increasing l the extremalizing value tends to fluctuate. For large l it should approach the value for the plane layer

$$R_H^* = 2/6^{\frac{1}{2}}, \tag{7.4}$$

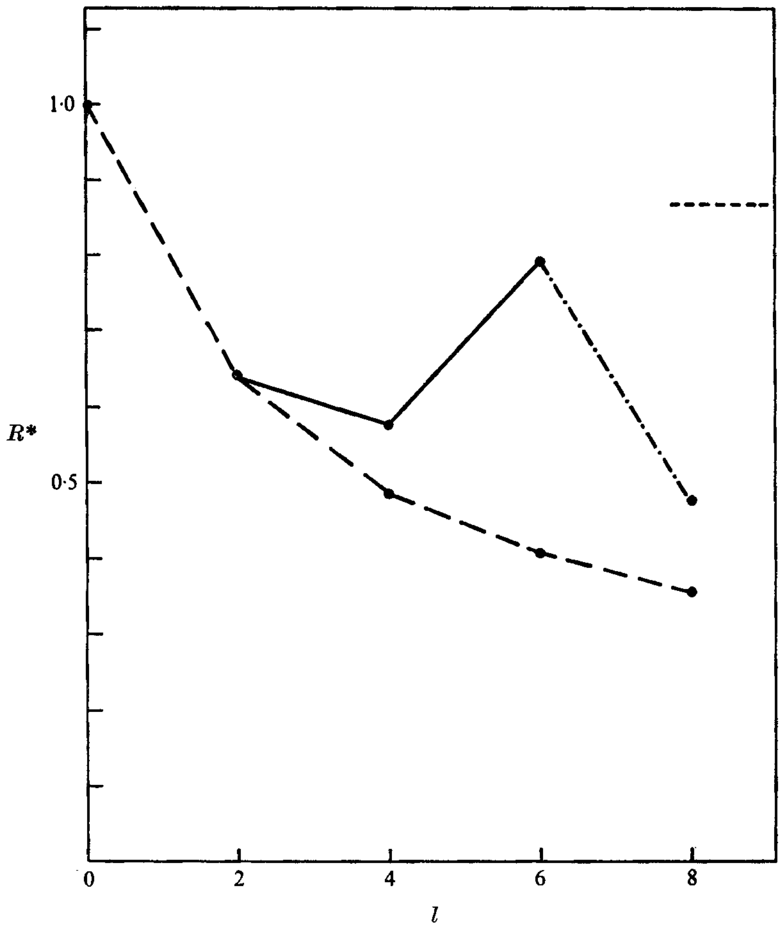

FIGURE 3. Value of R^* as a function of l. The lower dashed line connects values R_0^* corresponding to the axisymmetric solution; the upper dashed line indicates the value for hexagons in a plane layer.

which corresponds to the hexagon solution

$$w_H = (\tfrac{2}{3})^{\frac{1}{2}} \sum_{m=1}^{3} \cos \mathbf{k}_m . \mathbf{r},$$

where the vectors $\mathbf{k}$ satisfy the conditions

$$|\mathbf{k}_1| = |\mathbf{k}_2| = |\mathbf{k}_3| = a, \quad \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0.$$

Of course, a hexagonal pattern cannot be realized on a spherical surface according to Euler's theorem $V - E + F = 2$, where V, E and F denote the number of vertices, edges and faces, respectively, of the polyhedron. It seems, however, that a hexagonal pattern can be closely approximated, as the natural phenomenon displayed in figure 4 suggests.

Nearly hexagonal patterns corresponding to large values of l are also observed when a thin elastic spherical shell buckles under the influence of a uniform pressure. The corresponding mathematical problem has been treated analytically

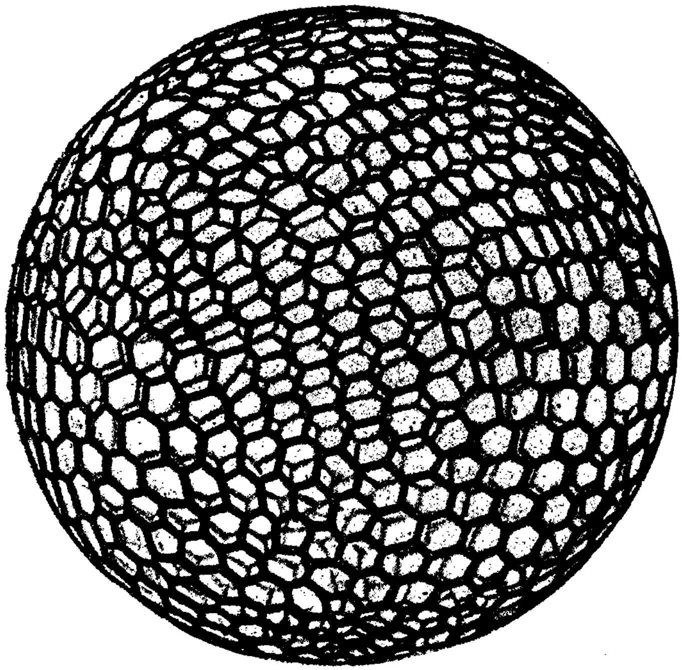

FIGURE 4. Skeleton of the radiolaria Aulonia hexagona, from Thompson (1942).

by Koiter (1969). In the limit of large l, Koiter derived asymptotic expressions for R^* for solutions (7.1 a, b) in the special case $n = \frac{1}{2}l$. However, the results show a decay of R^* proportional to $l^{-\frac{1}{4}}$ for large l. Hence it appears that the corresponding solutions do not approach a nearly uniform hexagonal pattern asymptotically. The problem of the correct asymptotic transition from the spherical case to the case of an infinite plane is part of the more general problem of the asymptotic transformation between spherical harmonics and trigonometric functions, which has not yet been solved in a uniform manner.

8. Discussion

The analysis of this paper is restricted to the problem of the *pattern* of convection in spherical shells. Since the radial dependence of the problem and the quantitative nature of the nonlinear terms do not enter the analysis the results apply to all problems with a degenerate set of eigenfunctions of the form (2.8 b) in the linear approximation. Some of the quantitative details, in particular higher-order effects, require specification of the radial dependence. Since this is beyond the scope of the present paper quantitative aspects will be mentioned only briefly in the following discussion.

A major result of our analysis is the fact that the solvability condition (2.15) does not resolve the degeneracy in the case of odd l. The coefficient $R^{(1)}$ vanishes for all spherical harmonics of odd order and so do the other coefficients $R^{(n)}$ with odd n, as we may deduce from the general nature of our argument. Since the contribution to R of the coefficients $R^{(n)}$ with even n is in general positive, we conclude that no subcritical instabilities occur in the case of odd l. In the case of even l, solutions satisfying (3.1) with a finite value of R^* can always exist at subcritical values of the Rayleigh number because of the freedom to choose the sign of ϵ such that $\epsilon R^{(1)}$ is negative.

Although the solvability condition (2.15) eliminates or at least reduces the degeneracy of solutions in the case of even l, there exist in general two or more solutions corresponding to different values of R^*. Only in the case $l = 2$ is a single solution, apart from rotational transformations, isolated by the solvability condition. In order to discuss the physical relevance of different values of R^* we must consider the dependence of the Rayleigh number on the amplitude ϵ, which for sufficiently small values of ϵ assumes the form

$$R = R^{(0)} + \epsilon R^{(1)} + \epsilon^2 R^{(2)}. \tag{8.1}$$

The minimum value of R according to this relation is given by

$$R_{\min} = R^{(0)} - |R^{(1)}|^2/2R^{(2)}. \tag{8.2}$$

$R^{(2)}$ is generally positive and mainly represents the effects of changes in the horizontally averaged temperature field induced by convection. Hence $R^{(2)}$ tends to vary little for different solutions at a given l. Neglecting those variations and neglecting the fact that higher-order contributions will modify the simple relation (8.2) we may conclude that solutions with the largest value of R^* correspond to the lowest value of the Rayleigh number at a given value of l. Since the solution with the lowest value $R_{\min}$ represents the physically realized one at least for low Rayleigh numbers, convection patterns with even l and an extreme value R^* will be preferred. Unless the relevant value $R^{(1)}(l)$ is very small and $R^{(0)}(l)$ attains a sharply defined minimum at an odd integer l we do not expect to see patterns with odd l realized. In most cases the minimizing value of l is of order r_0. In some cases it may be much lower, for example when the ratio between the thermal conductivity of the fluid and that of the boundary is high (Sparrow, Goldstein & Jonsson 1963). Hence the axisymmetric solution in the case $l = 2$ and the 'Platonic' solutions in the cases $l = 4$ and 6 may have physical significance even in the case of rather thin shells.

A complete discussion of the physically significant solutions requires a stability analysis, which cannot be done without specific information about the fluid shell. The results are expected to be similar, however, to the case of a plane layer, which has been treated in detail by Busse (1962, 1967). In particular, it may easily be seen from the analogous property of hexagons that all solutions are unstable for amplitudes $|\epsilon|$ below the value corresponding to the minimum Rayleigh number (8.2), since within this range the amplitude $|\epsilon|$ actually increases with decreasing Rayleigh number. Above this range the solution corresponding to the absolute minimum of (8.2) will be stable, at least for a finite range of amplitudes. In the case of a plane layer the extent of the stability region depends on the amount of asymmetric properties of the layer. We expect that the corresponding effect in the case of a fluid shell will be more persistent. Because the interior of the plane layer becomes more isothermal with increasing amplitude of convection the effect of temperature-dependent properties is minimized, and hexagonal cells are usually realized only in a limited range of Rayleigh numbers, and replaced by rolls beyond that range. If compressibility is taken into account the isothermal interior is replaced by one with an adiabatic temperature distribution and the preference for hexagonal convection based on temperature-dependent material properties will persist at high Rayleigh numbers. In the case of a spherical shell asymmetries persist even in the isothermal case and a competing solution as simple as the two-dimensional roll solution does not exist. This suggests that in most cases the results derived in this paper will hold qualitatively at much larger Rayleigh numbers than those for which expression (8.1) provides a good approximation. The forms of the lines of equal velocity in figures 1 and 2 are expected to change but not the symmetry of the patterns.

As we mentioned in the introduction, much of the recent interest in convection in a spherical system has been motivated by the problem of convection in the earth's mantle. Unfortunately, there is no convincing evidence that the results of this paper are applicable in this case. Gravity data and other global data related to mantle convection do not distinguish particular spherical harmonics and do not even show a preference for even orders. This indicates that either the inhomogeneous distribution of continents or inhomogeneities in the mantle exert a stronger influence than the nonlinearity of the dynamics, or that the basic mode of convection corresponds to the case $l = 2$ at the lower end of the spectrum.

9. Concluding remarks

The occurrence of convection in a spherically symmetric system is necessarily associated with the destruction of symmetry. In thinking about this problem it may be anticipated that convection patterns of the highest possible symmetry will be preferred. Attempts to use group theoretical methods in order to investigate convection patterns without using equations of motion have been made by Walzer (1971). Also, Spilhaus (1975) has suggested similarities between the pattern of plate tectonics and the symmetries of Platonic bodies. The analysis given in this paper indicates, however, that heuristic principles are not sufficient and that the dynamics of the system must be taken into account to infer the preferred pattern. Otherwise it cannot be understood why octahedron symmetry corresponds to a preferred pattern but tetrahedron symmetry does not.

The main result of this paper is the qualitative difference between convective patterns of odd and even order l. This difference does not have a direct analogue in the case of a plane layer, where solutions with different wavenumber differ only quantitatively. An analogy can be drawn, however, between spherical patterns of even order and the hexagon solution and its superpositions in the case of a plane layer, since both correspond to non-vanishing values of the functional (3.2). The functional vanishes for all other solutions for the plane layer such as rolls and rectangles, which are similar in this respect to spherical patterns of odd order.

Throughout the analysis of this paper it has been evident that the problem of degeneracy and its removal by nonlinear effects is common to all eigenvalue problems with spherical symmetry. The presence of inhomogeneities and physically distinguished directions in most phenomena account for the relative rarity of spherically homogeneous eigenvalue problems. Apart from spherical convection we have previously mentioned the problem of buckling of a spherical shell. Other examples are the cracking pattern of a symmetrically cooling solid sphere and oscillations of a non-rotating star. Finally, deviations from homogeneity on the cosmical scale may be related to the removal of degeneracy by nonlinear effects.

This work was first presented in 1974 at the Woods Hole Summer Program in Geophysical Fluid Dynamics under the directorship of Prof. G. Veronis. I am grateful to Prof. C. Lange for pointing out to me the work by Koiter and to Prof. E. Spiegel for referring me to the book *Symmetry* by H. Weyl. M. Bossen, E. Ivins and T. Virgil have assisted me in parts of the numerical calculations. The research was supported by the National Science Foundation under Grant GA-31247 and in part by NASA under Grant NSG 7002.

Note added in proof. I should like to use this opportunity to make two corrections to an earlier paper (*J. Fluid Mech.* 1972, **52**, 97–112). In expression (2.7), 4π should be replaced by 8π, and in expression (5.2), $2/3^{\frac{1}{2}}b$ by $(\frac{2}{3})^{\frac{1}{2}}/b$. These corrections do not affect any other parts of the paper. I am grateful to Dr Lipps for pointing out the second correction.

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